

Heronian Mean Operators and Their Application to Dual Hesitant Bipolar Fuzzy Multiple Attribute Decision Making

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Abstract In this paper, based on the advantages of bipolar fuzzy sets (BFSs) and hesitant fuzzy set (HFS), the concept of dual hesitant bipolar fuzzy sets (DHBFSs) is introduced and their operation rules and properties are discussed. The objective of this paper was to develop some novel Heronian Mean (HM) operators for any dual hesitant bipolar fuzzy elements (DHBFEs). The advantage of employing Heronian Mean operator mainly lie in considering the interrelationships between parameters and handling uncertainty and ambiguity information that exist in the real world. Thus, based on the

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DHBFS environment, we develop a family of dual hesitant bipolar fuzzy Heronian Mean aggregation operators, such as the generalized dual hesitant bipolar fuzzy Heronian Mean (GDHBFHM) operator, the generalized dual hesitant bipolar fuzzy weighted Heronian Mean (GDHBFWHM) operator, the dual hesitant bipolar fuzzy geometric Heronian Mean (DHBFGHM) operator, the dual hesitant bipolar fuzzy weighted geometric Heronian Mean (DHBFWGHM) operator. And the properties of these proposed operators are studied. Furthermore, an approach based on the GDHBFWHM operator and the DHBFWGHM operator is proposed for multiple attribute decision making problems under dual hesitant bipolar fuzzy environment. Finally, a numerical example and comparative analysis is given to illustrate the application of the proposed approach.

Keywords dual hesitant bipolar fuzzy sets; multiple attribute decision making; Heronian Mean operators; GDHBFWHM; DHBFWGHM

1 Introduction

Decision-making is prevalent in the realm of social development encompassing diverse activities like housing investments by residents, commodity purchases, tourism decisions, risk assessments for financial investments performance evaluations, and myriad other daily endeavors. These situations necessitate a scientific and holistic approach to decision-making. Given the escalating complexity of decision-making environments and the inherent fuzziness in experts' cognitive processes, fuzzy decision-making frameworks grounded in fuzzy set theory^[1] have garnered extensive research attention^[2–7]. As humanity's understanding of information fuzziness in real-world scenarios deepens, scholars have augmented and refined the fuzzy set theory, introducing advancements such as interval-valued fuzzy sets (IVFS)^[8], type-2 fuzzy sets (T2FS)^[9], and type-n fuzzy sets (TnFS)^[10]. Nevertheless, these frameworks predominantly focus on quantifying an element's belongingness to a set, leaving gaps in comprehensively portraying uncertain phenomena. Recognizing this limitation, Atanassov^[11] introduced the intuitionistic fuzzy set (IFS) in 1983. IFS embodies a dual dimensionality comprising both "membership" and "non-membership." The membership degree quantifies the extent to which an element belongs to a particular set, whereas the non-membership degree gauges the extent to which it does not. Importantly, IFS adheres to the principle that the sum of the membership and non-membership degrees must not exceed unity. However, in the real life, some situations are often encountered in which it is difficult to determine the membership of element to a given set due to the hesitation between a few different values. To solve this problem, recently, the concept of hesitant fuzzy set (HFS) was introduced by Torra^[12] to enrich the fuzzy set. It permits the membership of an element to a given set having a few different values, which has been a reasonable tool to describe the imprecise or uncertain decision information.

As a new generalized type of fuzzy sets, HFS has received a considerable attention and has been applied to a various field of decision making^[13–22]. More recently, the bipolar fuzzy set (BFS)^[23,24] has been proposed to depict uncertainty in multiple attribute decision making (MADM) problems, which use the positive membership degree and the negative membership degree to define an object. But different from the IFS, the range of membership degree of the

bipolar fuzzy set is $[-1, 1]$. In order to provide more comprehensive, accurate, and flexible decision-making information, Xu et al.^[25] proposed the concept of dual hesitant bipolar fuzzy set (DHBFS). It can simultaneously consider both positive and negative information as well as their hesitation, providing decision-makers with more comprehensive and detailed decision-making information. By introducing hesitation, dual hesitant bipolar sets can handle uncertain information more flexibly, enabling decision-makers to make more reasonable judgments when facing complex and ambiguous situations. In summary, by combining the characteristics of bipolar fuzzy sets and hesitant fuzzy sets, dual hesitant bipolar sets can more accurately reflect the actual situation, thereby improving the accuracy of decision-making. Jana et al.^[26] proposed bipolar fuzzy Dombi prioritized aggregation operators in multiple attribute decision making. Wei et al.^[27] proposed bipolar fuzzy Hamacher aggregation operators in multiple attribute decision making. Gao et al.^[28] proposed dual hesitant bipolar fuzzy Hamacher prioritized aggregation operators in multiple attribute decision making. Muhammad et al.^[29] proposed cubic bipolar fuzzy set with application to multi-criteria group decision making using geometric aggregation operators. Gao et al.^[30] proposed dual hesitant bipolar fuzzy hamacher aggregation operators and their applications to multiple attribute decision making.

As more and more scholars are becoming aware of the correlation between arguments, it is very important to the aggregated result. In order to better reflect the interaction between attributes and make the decision results more realistic, the Heronian Mean (HM)^[31] operator, which can effectively take the interrelationship between arguments into account, has drawn large quantity of scholars' attention. Tahir et al.^[32] proposed a novel approach towards Heronian Mean operators in multiple attribute decision making under the environment of bipolar complex fuzzy information. Qiyas et al.^[33] studied a novel approach of linguistic picture fuzzy Dombi Heronian Mean operators and their application to emergency program selection. Rao et al.^[34] discussed a novel MADM algorithm for physical education teaching quality evaluation based on 2-tuple linguistic neutrosophic numbers power Heronian Mean operators. Akram et al.^[35] extended MABAC method based on 2-tuple linguistic T-spherical fuzzy sets and Heronian Mean operators: An application to alternative fuel selection. Xu et al.^[36] proposed generalized cross weighted Heronian Mean operators and their application to binary classification. Anusha et al.^[37] proposed Einstein Heronian Mean aggregation operator and its application in decision making problems. Tian et al.^[38] developed covering-based compound mean operators arising from Heronian and Bonferroni Mean operators in fuzzy and intuitionistic fuzzy environments. Fan et al.^[39] discussed dynamic multi-attribute decision-making based on interval-valued picture fuzzy geometric Heronian Mean operators. Kumar et al.^[40] studied group decision making based on advanced intuitionistic fuzzy weighted Heronian Mean aggregation operator of intuitionistic fuzzy values. Hazwani et al.^[41] proposed Heronian Mean operators considering shapley fuzzy measure under interval neutrosophic vague environment for an investment decision. Tahir et al.^[42] proposed a method to multi-attribute decision making problems by using Heronian Mean operators based on linear diophantine uncertain linguistic settings. Zhang

et al.^[43] discussed spherical fuzzy Dombi power Heronian Mean aggregation operators for multiple attribute group decision-making. Wang et al.^[44] proposed interaction power Heronian Mean aggregation operators for multiple attribute decision making with T-spherical fuzzy information. Wang et al.^[45] developed a new three-way group decision-making model based on geometric Heronian Mean operators with q-rung orthopair uncertain linguistic information.

And it seems that there is no investigation on how to combine the Heronian Mean operations with dual hesitant bipolar fuzzy information. To do so, the motivations of this paper are outlined below:

1) Dealing with complex uncertainty: In the decision-making process, decision-makers often face complex and uncertain information. These pieces of information may contain both positive and negative evaluations, and the degree of evaluation may also be hesitant. In order to more effectively handle this complex uncertainty, researchers have proposed the concept of dual hesitant bipolar fuzzy sets (DHBFSs). On this basis, the further development of dual hesitant bipolar fuzzy Heronian Mean operators aims to provide a more refined and comprehensive method for integrating and processing such information.

2) Improving decision accuracy: Traditional fuzzy sets and hesitant fuzzy sets may have limitations in handling certain types of decision information. For example, they may not be able to fully capture the interactions or hesitations between attributes. Dual hesitant bipolar fuzzy Heronian Mean operators, by combining the characteristics of bipolar fuzzy sets and hesitant fuzzy sets, and utilizing the advantages of Heronian Mean, can more accurately reflect the actual situation, thereby improving the accuracy of decision-making.

3) Meeting practical application needs: In practical applications, decision makers may need to deal with problems that contain multiple types of uncertain information. For example, in fields such as multi-attribute decision-making, risk assessment, recommendation systems, etc., it may be necessary to consider both positive and negative evaluations as well as hesitation. The proposal of dual hesitant bipolar fuzzy Heronian Mean operators is aimed at meeting the needs of these practical applications.

The remainder of this paper is set out as follows. In Section 2, we introduce some basic concepts related to dual hesitant bipolar fuzzy sets (DHBFSs) and some operational laws of DHBFSs. In Section 3, we have developed some dual hesitant bipolar fuzzy Heronian Mean aggregation operators and studied some desirable properties of the proposed operators. In Section 4, we have applied these operators to develop some models for multiple attribute decision making (MADM) problems based on the Heronian Mean aggregation operators with dual hesitant bipolar fuzzy information. In Section 5, a practical example is given to verify the developed approach and to demonstrate its practicality and effectiveness. In Section 6, we conclude the paper and give some remarks.

2 Preliminaries

2.1 Dual Hesitant Bipolar Fuzzy Set (DHBFS)

Definition 1 Let X be a reference set, an HFS^[46] A on X is denoted by a function $h_A(x)$ that returns a subset of $[0, 1]$ when it is applied to X . To make it easier to understood, the HFS is expressed by a mathematical symbol:

$$A = \{\langle x, h_{A(x)} \rangle \mid x \in X\}, \quad (1)$$

where $h_{A(x)}$ is a set of some different values indicating all possible membership degrees of the element $x \in X$ to the set A . For convenience, $h_{A(x)}$ is called a hesitant fuzzy element (HFE), which is denoted by $h_{A(x)} = \{\gamma \mid \gamma \in h_{A(x)}\}$.

Definition 2 Let X be a fix set. A bipolar fuzzy set(BFS) is an object having the form:

$$B = \{\langle x, (u_B^+(x), v_B^-(x)) \rangle \mid x \in X\},$$

where the positive membership degree function $u_B^+(x): X \rightarrow [0, 1]$ denotes the satisfaction degree of an element x to the property corresponding to a BFSB and the negative membership degree function $v_B^-(x): X \rightarrow [0, 1]$ denotes satisfaction degree of an element x to some implicit counter-property corresponding to a BFSB, respectively, and, for every $x \in X$. For convenience, let $b = (u^+, v^-)$ be a bipolar fuzzy number (BFN).

Definition 3 Let $b = (u^+, v^+)$, $b_1 = (u_1^+, v_1^-)$ and $b_2 = (u_2^+, v_2^-)$ be three BFNs, and some basic operations on them are defined as follows:

- (1) $b_1 \oplus b_2 = (u_1^+ + u_2^+ - u_1^+ u_2^+, -|v_1^-||v_2^-|)$;
- (2) $b_1 \otimes b_2 = (u_1^+ u_2^+, v_1^- + v_2^- - v_1^- v_2^-)$;
- (3) $\lambda b = (1 - (1 - u^+)^{\lambda}, -|v^-|^{\lambda}), \lambda > 0$;
- (4) $b^{\lambda} = ((u^+)^{\lambda}, -1 + |1 + v^-|^{\lambda}), \lambda > 0$.

Definition 4 Given a fixed set X , then a hesitant bipolar fuzzy set (HBFS) on X is in terms of a function that when applied to X returns a subset of $[0, 1]$. The dual hesitant bipolar fuzzy set (DHBFS) can be expressed by mathematical symbol:

$$P = \{\langle x, h_{P(x)} \rangle \mid x \in X\}, \quad (2)$$

where $h_{P(x)}$ is a set of some bipolar values in P , $h_{P(x)} = (u^+, v^-)$, and the possible membership degrees $u^+: X \rightarrow [0, 1]$ denotes the possible satisfaction degree of an element x to the property corresponding to a DHBFS and the possible nonmembership degrees $u^-: X \rightarrow [-1, 0]$ denotes the possible satisfaction degree of an element x to some implicit counter property corresponding to a DHBFS, respectively, and, for every $x \in X$, with the conditions: $0 \leq u^+ \leq 1, -1 \leq v^- \leq 0$.

For convenience, we called $p = \langle \{u^+\}, \{v^-\} \rangle$ a dual hesitant bioplar fuzzy element (DHBFE).

Definition 5 Let $p = \langle \{u^+\}, \{v^-\} \rangle$, $p_1 = \langle \{u_1^+\}, \{v_1^-\} \rangle$ and $p_2 = \langle \{u_2^+\}, \{v_2^-\} \rangle$ be three DHBFEs, and some basic operations on them are defined as follows:

- (1) $p_1 \oplus p_2 = (\{u_1^+ + u_2^+ - u_1^+ u_2^+\}, \{-|v_1^-||v_2^-|\})$;

- (2) $p_1 \otimes p_2 = (\{u_1^+ u_2^+\}, \{v_1^- + v_2^- - v_1^- v_2^-\})$;
 (3) $\lambda p = (\{1 - (1 - u^+)^{\lambda}\}, \{-|v^-|^{\lambda}\}), \lambda > 0$;
 (4) $p^{\lambda} = (\{(u^+)^{\lambda}\}, \{-1 + |1 + v^-|^{\lambda}\}), \lambda > 0$.

Definition 6 Let $p = \langle u^+, v^- \rangle$ be a DHBFE, then we define the score function of p as follows:

$$S(p) = \frac{1}{2} \left(1 + \frac{1}{\#u^+} \sum_{\gamma^+ \in u^+} \gamma^+ + \frac{1}{\#v^-} \sum_{\eta^- \in v^-} \eta^- \right), \quad (3)$$

where $\#u^+$ and $\#v^-$ are the numbers of the elements in u^+ and v^- respectively.

Definition 7 Let $p = \langle u^+, v^- \rangle$ be a DHBFE, then we define the accuracy function of p as follows:

$$H(p) = \frac{1}{2} \left(\frac{1}{\#u^+} \sum_{\gamma^+ \in u^+} \gamma^+ - \frac{1}{\#v^-} \sum_{\eta^- \in v^-} \eta^- \right), \quad (4)$$

where $\#u^+$ and $\#v^-$ are the numbers of the elements in u^+ and v^- respectively.

Inspired by this idea, based on the score $S(p)$ and the accuracy degree $H(p)$, we can compare and rank two DHBFEs in the following definition.

Definition 8 Let $p_1 = \langle \{u_1^+\}, \{v_1^-\} \rangle$ and $p_2 = \langle \{u_2^+\}, \{v_2^-\} \rangle$ be two HBFEs, $S(p_1)$ be the score function of p_1 and $S(p_2)$ be the score function of p_2 , $H(p_1)$ be the accuracy function of p_1 and $H(p_2)$ be the accuracy function of p_2 . Then

- (1) If $S(p_1) < S(p_2)$, then $p_1 < p_2$;
 (2) If $S(p_1) > S(p_2)$, then $p_1 > p_2$;
 (3) If $S(p_1) = S(p_2)$, then
 If $H(p_1) = H(p_2)$, then $p_1 \sim p_2$;
 If $H(p_1) > H(p_2)$, then $p_1 \succ p_2$.

2.2 HM Operator

The HM^[31] is an important modeling tool to reflect the interrelationship between the input arguments, which is defined as follows.

Definition 9 Let $(h_1, h_2, \dots, h_n)$ be a collection of nonnegative real number so that $h_i \in [0, 1]$ ($i = 1, 2, \dots, n$), then the HM is defined as follows:

$$\text{HM}(h_1, h_2, \dots, h_n) = \frac{2}{n(n+1)} \sum_{k=1}^n \sum_{t=k}^n \sqrt{h_k h_t}. \quad (5)$$

Based on Definition 9, Yu^[47] proposed the generalized Heronian Mean as follows:

Definition 10 Let $\rho, \psi \geq 0$ with $\rho + \psi > 0$, $(h_1, h_2, \dots, h_n)$ be a collection of nonnegative real number so that $h_i \in [0, 1]$ ($i = 1, 2, \dots, n$), if

$$\text{GHM}^{\rho, \psi}(h_1, h_2, \dots, h_n) = \left(\frac{2}{n(n+1)} \sum_{k=1}^n \sum_{t=k}^n h_k^{\rho} h_t^{\psi} \right)^{\frac{1}{\rho + \psi}}, \quad (6)$$

then $\text{HM}^{\rho, \psi}$ is called the generalized Heronian Mean operator or Heronian Mean operator with parameter. It is obvious that when $\rho = \psi = \frac{1}{2}$, the $\text{HM}^{\rho, \psi}$ operator transforms into the above-mentioned traditional HM operator.

3 Dual Hesitant Bipolar Fuzzy HM Aggregation Operators

3.1 Dual Hesitant Bipolar Fuzzy HM Aggregation Operators

Definition 11 Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs, the generalized dual hesitant bipolar fuzzy Heronian Mean (GDHBFHM) operator is a mapping DHBFBM: $\text{DHBFE}^n \rightarrow \text{DHBFE}$ can be defined as

$$\text{GDHBFHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) = \left(\frac{2}{n(n+1)} \bigoplus_{i=1}^n \bigoplus_{j=i}^n (p_i^\rho \otimes p_j^\psi) \right)^{\frac{1}{\rho+\psi}}, \quad (7)$$

where $\rho, \psi \geq 0$.

Theorem 1 Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs, then the aggregation result using DHBFBM operator is also a DHBFE and

$$\begin{aligned} & \text{GDHBFHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) \\ &= \left\langle \left\{ \left(1 - \prod_{i=1}^n (1 - (\gamma_i^+)^{\rho} (\gamma_j^-)^{\psi})^{\frac{2}{n(n+1)}} \right)^{\frac{1}{\rho+\psi}} \right\}, \left\{ -1 + \left(1 - \prod_{i=1}^n (1 - (1 + \eta_i^-)^{\rho} (1 + \eta_j^-)^{\psi})^{\frac{2}{n(n+1)}} \right)^{\frac{1}{\rho+\psi}} \right\} \right\rangle, \quad (8) \end{aligned}$$

where $\rho, \psi \geq 0$.

Proof. Based on the operational laws of DHBFEs, we have

$$p_i^\rho = \langle \{(\gamma_i^+)^{\rho}\}, \{-1 + (1 + \eta_i^-)^{\rho}\} \rangle, p_j^\psi = \langle \{(\gamma_j^+)^{\psi}\}, \{-1 + (1 + \eta_j^-)^{\psi}\} \rangle.$$

Thus

$$p_i^\rho \otimes p_j^\psi = \langle \{(\gamma_i^+)^{\rho} (\gamma_j^+)^{\psi}\}, \{-1 + (1 + v_i^-)^{\rho} (1 + v_j^-)^{\psi}\} \rangle$$

Therefore,

$$\bigoplus_{i=1}^n \bigoplus_{j=i}^n (p_i^\rho \otimes p_j^\psi) = \left\langle \left\{ 1 - \prod_{i=1, j=i}^n (1 - (\gamma_i^+)^{\rho} (\gamma_j^+)^{\psi}) \right\}, \left\{ \prod_{i=1, j=i}^n (1 - (1 + v_i^-)^{\rho} (1 + v_j^-)^{\psi}) \right\} \right\rangle.$$

Furthermore

$$\frac{2}{n(n+1)} \bigoplus_{i=1}^n \bigoplus_{j=i}^n (p_i^\rho \otimes p_j^\psi) \left\langle \left\{ 1 - \prod_{i=1, j=i}^n (1 - (\gamma_i^+)^{\rho} (\gamma_j^+)^{\psi})^{\frac{2}{n(n+1)}} \right\}, \left\{ \prod_{i=1, j=i}^n (1 - (1 + v_i^-)^{\rho} (1 + v_j^-)^{\psi})^{\frac{2}{n(n+1)}} \right\} \right\rangle.$$

Therefore,

$$\begin{aligned} & \text{GDHBFHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) = \left(\frac{2}{n(n+1)} \bigoplus_{i=1}^n \bigoplus_{j=i}^n (p_i^\rho \otimes p_j^\psi) \right)^{\frac{1}{\rho+\psi}} \\ &= \left\langle \left\{ \left(1 - \prod_{i=1}^n (1 - (\gamma_i^+)^{\rho} (\gamma_j^+)^{\psi})^{\frac{2}{n(n+1)}} \right)^{\frac{1}{\rho+\psi}} \right\}, \left\{ -1 + \left(1 - \prod_{i=1}^n (1 - (1 + \eta_i^-)^{\rho} (1 + \eta_j^-)^{\psi})^{\frac{2}{n(n+1)}} \right)^{\frac{1}{\rho+\psi}} \right\} \right\rangle. \end{aligned}$$

Then, we complete the proof.

In what follows, we shall investigate some desirable properties of GDHBFHM.

Theorem 2 (Idempotency) Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs. If all p_i ($i = 1, 2, \dots, n$) are equal, i.e., $p_i = p = \langle \{\gamma^+\}, \{\eta^-\} \rangle$ for all i , then

$$\text{GDHBFHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) = \langle \{\gamma^+\}, \{\eta^-\} \rangle. \quad (9)$$

Theorem 3 (Monotonicity) Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) and $p'_i = (u_i'^+, v_i'^-)$ ($i = 1, 2, \dots, n$) be two collections of DHBFEs. If $p_i \leq p'_i$ for all i , then

$$\text{GDHBFHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) \leq \text{GDHBFHM}^{\rho, \psi}(p'_1, p'_2, \dots, p'_n). \quad (10)$$

Theorem 4 (Boundedness) Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs, and let $p^- = \min_i p_i, p^+ = \max_i p_i$, then

$$p^- \leq \text{GDHBFHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) \leq p^+. \quad (11)$$

Now, we can discuss some special cases of the GDHBFHM operator with respect to the parameter ρ, ψ .

Case 1: If $\rho = \psi = \frac{1}{2}$, then the GDHBFHM reduces to DHBFHM as follows:

$$\begin{aligned} \text{GDHBFHM}^{\frac{1}{2}, \frac{1}{2}}(p_1, p_2, \dots, p_n) &= \left(\frac{2}{n(n+1)} \bigoplus_{i=1}^n \bigoplus_{j=i}^n (p_i^{1/2} \otimes p_j^{1/2}) \right) \\ &= \left\langle \left\{ \left(1 - \prod_{i=1}^n \left(1 - \sqrt{\gamma_i^+ \gamma_j^+} \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{\rho+\psi}} \right\}, \left\{ -1 + \left(1 - \prod_{i=1}^n \left(1 - \sqrt{(1+\eta_i^-)(1+\eta_j^-)} \right)^{\frac{2}{n(n+1)}} \right) \right\} \right\rangle. \end{aligned} \quad (12)$$

Case 2: If $\psi \rightarrow 0$, then the GDHBFHM reduces to the generalized dual hesitant bipolar fuzzy mean (GDHBFM) as follows:

$$\begin{aligned} \lim_{\psi \rightarrow 0} \text{GDHBFHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) &= \left(\frac{1}{n} \bigoplus_{i=1}^n (p_i^\rho) \right)^{\frac{1}{\rho}} \\ &= \left\langle \left\{ \left(1 - \prod_{i=1}^n \left(1 - (\gamma_i^+)^{\rho} \right)^{\frac{1}{n}} \right)^{\frac{1}{\rho}} \right\}, \left\{ -1 + \left(1 - \prod_{i=1}^n \left(1 - (1 + \eta_i^-)^{\rho} \right)^{\frac{1}{n}} \right)^{\frac{1}{\rho}} \right\} \right\rangle. \end{aligned} \quad (13)$$

Case 3: If $\rho = 2, \psi \rightarrow 0$, then the GDHBFHM reduces to the generalized dual hesitant bipolar fuzzy square mean (GDHBFMSM) as follows:

$$\begin{aligned} \text{GDHBFHM}^{2,0}(p_1, p_2, \dots, p_n) &= \left(\frac{1}{n} \bigoplus_{i=1}^n (p_i^2) \right)^{\frac{1}{2}} \\ &= \left\langle \left\{ \left(1 - \prod_{i=1}^n \left(1 - (\gamma_i^+)^2 \right)^{\frac{1}{n}} \right)^{\frac{1}{2}} \right\}, \left\{ -1 + \left(1 - \prod_{i=1}^n \left(1 - (1 + \eta_i^-)^2 \right)^{\frac{1}{n}} \right)^{\frac{1}{2}} \right\} \right\rangle. \end{aligned} \quad (14)$$

Case 4: If $\rho = 1, \psi \rightarrow 0$, then the GDHBFHM reduces to the dual hesitant bipolar fuzzy average (DHBFA) as follows:

$$\begin{aligned} \text{GDHBFHM}^{1,0}(p_1, p_2, \dots, p_n) &= \frac{1}{n} \bigoplus_{i=1}^n (p_i) \\ &= \left\langle \left\{ 1 - \prod_{i=1}^n (1 - \gamma_i^+)^{\frac{1}{n}} \right\}, \left\{ -1 + \left(1 - \prod_{i=1}^n (-\eta_i^-)^{\frac{1}{n}} \right) \right\} \right\rangle. \end{aligned} \quad (15)$$

Case 5: If $\rho = \psi = 1$, then the GDHBFHM reduces to the dual hesitant bipolar fuzzy interrelated square mean as follows:

$$\begin{aligned} \text{GDHBFHM}^{1,1}(p_1, p_2, \dots, p_n) &= \left(\frac{2}{n(n+1)} \bigoplus_{i=1}^n \bigoplus_{j=i}^n (p_i \otimes p_j) \right)^{\frac{1}{2}} \\ &= \left\langle \left\{ \left(1 - \prod_{i=1}^n \left(1 - \gamma_i^+ \gamma_j^+ \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{2}} \right\}, \left\{ -1 + \left(1 - \prod_{i=1}^n \left(1 - (1 + \eta_i^-) (1 + \eta_j^-) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{2}} \right\} \right\rangle. \end{aligned} \quad (16)$$

Case 6: If $\rho \rightarrow \infty, \psi = 0$, then the GDHBFHM reduces to

$$\lim_{\rho \rightarrow \infty} \text{GDHBFHM}^{\rho,0}(p_1, p_2, \dots, p_n) = \lim_{\rho \rightarrow \infty} \left(\frac{1}{n} \bigoplus_{i=1}^n (p_i^\rho) \right)^{\frac{1}{\rho}} = \max_i (p_i). \quad (17)$$

Case 7: If $\rho \rightarrow 0, \psi = 0$, then the GDHBFHM reduces to

$$\lim_{\rho \rightarrow 0} \text{GDHBFHM}^{\rho,0}(p_1, p_2, \dots, p_n) = \lim_{\rho \rightarrow 0} \left(\frac{1}{n} \bigoplus_{i=1}^n (p_i^\rho) \right)^{\frac{1}{\rho}} = \bigotimes_{i=1}^n (p_i)^{\frac{1}{n}} = \left(\bigotimes_{i=1}^n p_i \right)^{\frac{1}{n}}. \quad (18)$$

3.2 Dual Hesitant Bipolar Fuzzy WHM Aggregation Operators

In the above section, the GDHBFHM operator does not consider the importance of the aggregated arguments. However, in many real practical situations, especially in MADM, the weights of attributes play an important role in the process of aggregation. To overcome the limitation of GDHBFHM, we shall propose the generalized dual hesitant bipolar fuzzy weighted Heronian Mean (GDHBFWHM) operator as follows.

Definition 12 Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs, $\mathbf{w} = (w_1, w_2, \dots, w_n)^T$ be the weight vector of $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$), and $w_j > 0, \sum_{j=1}^n w_j = 1$, the generalized dual hesitant bipolar fuzzy weighted Heronian Mean (GDHBFWHM) operator is a mapping GDHBFWHM:

DHBFEⁿ → DHBFE can be defined as

$$\text{GDHBFWHM}^{\rho,\psi}(p_1, p_2, \dots, p_n) = \left(\bigoplus_{i=1}^n \bigoplus_{j=i}^n \left(w_i w_j (p_i^\rho \otimes p_j^\psi) \right) \right)^{\frac{1}{\rho+\psi}}, \quad (19)$$

where $\rho, \psi \geq 0$.

Theorem 5 Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs, then the aggregation result using GDHBFWM operator is also a DHBFE and

$$\text{GDHBFWM}^{\rho, \psi}(p_1, p_2, \dots, p_n) = \left\langle \left(1 - \prod_{i=1, j=i}^n (1 - (\gamma_i^+)^{\rho} (\gamma_j^+)^{\psi})^{w_i w_j} \right)^{\frac{1}{\rho + \psi}}, \left(-1 + \left(1 - \prod_{i=1, j=i}^n (1 - (1 + \eta_i^-)^{\rho} (1 + \eta_j^-)^{\psi})^{w_i w_j} \right)^{\frac{1}{\rho + \psi}} \right) \right\rangle. \quad (20)$$

Proof. Based on the operational laws of DHBFEs, we have

$$p_i^{\rho} = \langle \{(\gamma_i^+)^{\rho}\}, \{-1 + (1 + \eta_i^-)^{\rho}\} \rangle, p_j^{\psi} = \langle \{(\gamma_j^+)^{\psi}\}, \{-1 + (1 + \eta_j^-)^{\psi}\} \rangle.$$

Thus

$$p_i^{\rho} \otimes p_j^{\psi} = \langle \{(\gamma_i^+)^{\rho} (\gamma_j^+)^{\psi}\}, \{-1 + (1 + \eta_i^-)^{\rho} (1 + \eta_j^-)^{\psi}\} \rangle.$$

Therefore,

$$\bigoplus_{i=1}^n \bigoplus_{j=i}^n ((w_i p_i^{\rho}) \otimes (w_j p_j^{\psi})) = \left\langle \left\{ 1 - \prod_{i=1, j=i}^n (1 - (\gamma_i^+)^{\rho} (\gamma_j^+)^{\psi})^{w_i w_j} \right\}, \left\{ -1 + \prod_{i=1, j=i}^n (1 - (1 + \eta_i^-)^{\rho} (1 + \eta_j^-)^{\psi})^{w_i w_j} \right\} \right\rangle.$$

Therefore,

$$\begin{aligned} \text{GDHBFWM}^{\rho, \psi}(p_1, p_2, \dots, p_n) &= \left(\bigoplus_{i=1}^n \bigoplus_{j=i}^n (w_i w_j (p_i^{\rho} \otimes p_j^{\psi})) \right)^{\frac{1}{\rho + \psi}} \\ &= \left\langle \left\{ \left(1 - \prod_{i=1, j=i}^n (1 - (\gamma_i^+)^{\rho} (\gamma_j^+)^{\psi})^{w_i w_j} \right)^{\frac{1}{\rho + \psi}} \right\}, \left\{ -1 + \left(1 - \prod_{i=1, j=i}^n (1 - (1 + \eta_i^-)^{\rho} (1 + \eta_j^-)^{\psi})^{w_i w_j} \right)^{\frac{1}{\rho + \psi}} \right\} \right\rangle. \end{aligned}$$

Then, we complete the proof.

It can be easily proved that the GDHBFWM operator has the following properties.

Theorem 6 (Idempotency) Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs. If all p_i ($i = 1, 2, \dots, n$) are equal, i.e., $p_i = p = \langle \{\gamma^+\}, \{\eta^-\} \rangle$ for all i , then

$$\text{GDHBFWM}^{\rho, \psi}(p_1, p_2, \dots, p_n) = \langle \{\gamma^+\}, \{\eta^-\} \rangle. \quad (21)$$

Theorem 7 (Monotonicity) Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) and $p'_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be two collections of DHBFEs. If $p_i \leq p'_i$ for all i , then

$$\text{GDHBFWM}^{\rho, \psi}(p_1, p_2, \dots, p_n) \leq \text{GDHBFWM}^{\rho, \psi}(p'_1, p'_2, \dots, p'_n). \quad (22)$$

Theorem 8 (Boundedness) Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs, and let $p^- = \min_i p_i, p^+ = \max_i p_i$, then

$$p^- \leq \text{GDHBFWM}^{\rho, \psi}(p_1, p_2, \dots, p_n) \leq p^+. \quad (23)$$

3.3 Dual Hesitant Bipolar Fuzzy GHM Aggregation Operators

In the following, Yu^[41] explored the geometric Heronian Mean (GHM) considering both the HM and the geometric mean (GM).

Definition 13 Let $\rho, \psi \geq 0$ with $\rho + \psi > 0$, $(h_1, h_2, \dots, h_n)$ be a collection of nonnegative real number, if

$$\text{HM}^{\rho, \psi}(h_1, h_2, \dots, h_n) = \frac{1}{\rho + \psi} \left(\prod_{k=1, t=k}^n (\rho h_k + \psi h_t) \right)^{\frac{2}{n(n+1)}}, \quad (24)$$

then $\text{HM}^{\rho, \psi}$ is called the geometric Heronian Mean (GHM) operator.

The geometric Heronian Mean (GHM) operator^[41], however, have usually been used in situations where the input arguments are the non-negative real numbers. In this section, we shall investigate the GHM operator with DHBFEs. Based on Definition 13, we give the definition of the dual hesitant bipolar fuzzy geometric Heronian Mean (DHBFGHM) operator as follows.

Definition 14 Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs, the DHBFGHM operator is a mapping:

$\text{DHBFE}^n \rightarrow \text{DHBFE}$ can be defined as

$$\text{DHBFGHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) = \frac{1}{\rho + \psi} \left(\bigotimes_{i=1}^n \bigotimes_{j=i}^n (\rho p_i \oplus \psi p_j) \right)^{\frac{2}{n(n+1)}}, \quad (25)$$

where $\rho, \psi \geq 0$.

Theorem 9 Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs, then the aggregation result using DHBFGHM operator is also a DHBFE and

$$\begin{aligned} & \text{DHBFGHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) \\ &= \left\langle \left\{ 1 - \left(1 - \prod_{i=1}^n (1 - (1 - \gamma_i^+)^{\rho} (1 - \gamma_j^+)^{\psi})^{\frac{2}{n(n+1)}} \right)^{\frac{1}{\rho + \psi}} \right\}, \left\{ - \left(1 - \prod_{i=1}^n (1 - (-\eta_i^-)^{\rho} (-\eta_j^-)^{\psi})^{\frac{2}{n(n+1)}} \right)^{\frac{1}{\rho + \psi}} \right\} \right\rangle, \quad (26) \end{aligned}$$

where $\rho, \psi \geq 0$.

Proof. Based on the operational laws of DHBFEs, we have

$$\rho p_i = \left\langle \left\{ \sqrt[n]{1 - (1 - u_i^q)^{\rho}} \right\}, \{v_i^{\rho}\} \right\rangle, \psi p_j = \left\langle \left\{ \sqrt[n]{1 - (1 - u_j^q)^{\psi}} \right\}, \{v_j^{\psi}\} \right\rangle.$$

Thus

$$\rho p_i \oplus \psi p_j = \left\langle \{1 - (1 - \gamma_i^+)^{\rho} (1 - \gamma_j^+)^{\psi}\}, \{(-\eta_i^-)^{\rho} (-\eta_j^-)^{\psi}\} \right\rangle.$$

Therefore,

$$\bigotimes_{i=1}^n \bigotimes_{j=i}^n (\rho p_i \oplus \psi p_j) = \left\langle \left\{ \prod_{i=1, j=i}^n (1 - (1 - \gamma_i^+)^{\rho} (1 - \gamma_j^+)^{\psi}) \right\}, \left\{ -1 + \prod_{i=1, j=i}^n (1 - (-\eta_i^-)^{\rho} (-\eta_j^-)^{\psi}) \right\} \right\rangle.$$

Furthermore

$$\bigotimes_{i=1}^n \bigotimes_{j=i}^n (\rho p_i \oplus \psi p_j)^{\frac{2}{n(n+1)}} = \left\langle \left\{ \prod_{i=1, j=i}^n (1 - (1 - \gamma_i^+)^{\rho} (1 - \gamma_j^+)^{\psi})^{\frac{2}{n(n+1)}} \right\}, \left\{ -1 + \prod_{i=1, j=i}^n (1 - (-\eta_i^-)^{\rho} (-\eta_j^-)^{\psi})^{\frac{2}{n(n+1)}} \right\} \right\rangle.$$

Therefore,

$$\begin{aligned} \text{DHBFGHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) &= \frac{1}{\rho + \psi} \left(\bigotimes_{i=1}^n \bigotimes_{j=i}^n ((\rho p_i \oplus \psi p_j)) \right)^{\frac{2}{n(n+2)}} \\ &= \left\langle \left\{ 1 - \left(1 - \prod_{i=1}^n (1 - (1 - \gamma_i^+)^{\rho} (1 - \gamma_j^+)^{\psi})^{\frac{2}{n(n+1)}} \right)^{\frac{1}{\rho + \psi}} \right\}, \left\{ - \left(1 - \prod_{i=1}^n (1 - (-\eta_i^-)^{\rho} (-\eta_j^-)^{\psi})^{\frac{2}{n(n+1)}} \right)^{\frac{1}{\rho + \psi}} \right\} \right\rangle. \end{aligned}$$

Then, we complete the proof.

In what follows, we shall investigate some desirable properties of DHBFGHM.

Theorem 10 (Idempotency) Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs. If all p_i ($i = 1, 2, \dots, n$) are equal, i.e., $p_i = p = \langle \{\gamma^+\}, \{\eta^-\} \rangle$ for all i , then

$$Hq - \text{DHBFGHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) = \langle \{\gamma^+\}, \{\eta^-\} \rangle. \quad (27)$$

Theorem 11 (Monotonicity) Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) and $p'_i = (u_i'^+, v_i'^-)$ ($i = 1, 2, \dots, n$) be two collections of DHBFEs. If $p_i \leq p'_i$ for all i , then

$$\text{DHBFGHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) \leq \text{DHBFGHM}^{\rho, \psi}(p'_1, p'_2, \dots, p'_n). \quad (28)$$

Theorem 12 (Boundedness) Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs, and let $p^- = \min_i p_i, p^+ = \max_i p_i$, then

$$p^- \leq \text{DHBFGHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) \leq p^+. \quad (29)$$

3.4 Dual Hesitant Bipolar Fuzzy WGHM Aggregation Operators

In the above section, the DHBFGHM operator doesn't consider the importance of the aggregated arguments. However, in many real practical situations, especially in MADM, the weights of attributes play an important role in the process of aggregation. To overcome the limitation of DHBFGHM, we shall propose the dual hesitant bipolar fuzzy weighted geometric Heronian Mean (DHBFWGHM) operator as follows.

Definition 15 Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs, $\mathbf{w} = (w_1, w_2, \dots, w_n)^T$ be the weight vector of $\mathbf{p}_i = \langle g_i, h_i \rangle$ ($i = 1, 2, \dots, n$), and $w_j > 0, \sum_{j=1}^n w_j = 1$, the DHBFWGHM operator is a mapping DHBFWGHM:

DHBFEⁿ → DHBFE can be defined as

$$\text{DHBFWGHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) = \frac{1}{\rho + \psi} \left(\bigotimes_{i=1}^n \bigotimes_{j=i}^n (\rho p_i \oplus \psi p_j)^{w_i w_j} \right), \quad (30)$$

where $\rho, \psi \geq 0$.

Theorem 13 Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs, then the aggregation result using DHBFWGHM operator is also a DHBFE and

$$\begin{aligned} \text{DHBFWGHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) &= \left\langle \left\{ 1 - \left(1 - \prod_{i=1}^n (1 - (1 - \gamma_i^+)^{\rho} (1 - \gamma_j^+)^{\psi})^{w_i w_j} \right)^{\frac{1}{\rho + \psi}} \right\}, \left\{ - \left(1 - \prod_{i=1}^n (1 - (-\eta_i^-)^{\rho} (-\eta_j^-)^{\psi})^{w_i w_j} \right)^{\frac{1}{\rho + \psi}} \right\} \right\rangle, \end{aligned} \quad (31)$$

where $\rho, \psi \geq 0$.

Proof. Based on the operational laws of DHBFEs, we have

$$\rho p_i = \langle \{1 - (1 - \gamma_i^+)^{\rho}\}, \{(-\eta_i^-)^{\rho}\} \rangle, \psi p_j = \langle \{1 - (1 - \gamma_j^+)^{\psi}\}, \{(-v_j^-)^{\psi}\} \rangle.$$

Thus

$$\rho p_i \oplus \psi p_j = \langle \{1 - (1 - \gamma_i^+)^{\rho}(1 - \gamma_j^+)^{\psi}\}, \{(-\eta_i^-)^{\rho}(-\eta_j^-)^{\psi}\} \rangle.$$

Furthermore,

$$(\rho p_i \oplus \psi p_j)^{w_i w_j} = \langle \{(1 - (1 - \gamma_i^+)^{\rho}(1 - \gamma_j^+)^{\psi})^{w_i w_j}\}, \{-1 + (1 - (-\eta_i^-)^{\rho}(-\eta_j^-)^{\psi})^{w_i w_j}\} \rangle.$$

Therefore,

$$\bigotimes_{i=1}^n \bigotimes_{j=i}^n (\rho p_i \oplus \psi p_j)^{w_i w_j} = \left\langle \left\{ \left(\prod_{i=1, j=i}^n (1 - (1 - \gamma_i^+)^{\rho}(1 - \gamma_j^+)^{\psi}) \right)^{w_i w_j} \right\}, \left\{ -1 + \prod_{i=1, j=i}^n (1 - (-\eta_i^-)^{\rho}(-\eta_j^-)^{\psi})^{w_i w_j} \right\} \right\rangle.$$

Furthermore

$$\bigotimes_{i=1}^n \bigotimes_{j=i}^n (\rho p_i \oplus \psi p_j)^{\frac{2}{n(n+1)}} = \left\langle \left\{ \prod_{i=1, j=i}^n (1 - (1 - \gamma_i^+)^{\rho}(1 - \gamma_j^+)^{\psi})^{\frac{2}{n(n+1)}} \right\}, \left\{ -1 + \prod_{i=1, j=i}^n (1 - (-\eta_i^-)^{\rho}(-\eta_j^-)^{\psi})^{\frac{2}{n(n+1)}} \right\} \right\rangle.$$

Therefore,

$$\begin{aligned} Hq - \text{ROFWGHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) &= \frac{1}{\rho + \psi} \left(\bigotimes_{i=1}^n \bigotimes_{j=i}^n (\rho p_i \oplus \psi p_j) \right)^{w_i w_j} \\ &= \left\langle \left\{ 1 - \left(1 - \prod_{i=1, j=i}^n (1 - (1 - \gamma_i^+)^{\rho}(1 - \gamma_j^+)^{\psi})^{w_i w_j} \right)^{\frac{1}{\rho + \psi}} \right\}, \left\{ - \left(1 - \prod_{i=1, j=i}^n (1 - (-\eta_i^-)^{\rho}(-\eta_j^-)^{\psi})^{w_i w_j} \right)^{\frac{1}{\rho + \psi}} \right\} \right\rangle. \end{aligned}$$

Then, we complete the proof.

In what follows, we shall investigate some desirable properties of DHBFWGHM.

Theorem 14 (Idempotency) Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs. If all p_i ($i = 1, 2, \dots, n$) are equal, i.e., $p_i = p = \langle \{\gamma^+\}, \{\eta^-\} \rangle$ for all i , then

$$\text{DHBFWGHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) = \langle \{\gamma^+\}, \{\eta^-\} \rangle. \quad (32)$$

Theorem 15 (Monotonicity) Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) and $p'_i = (u_i'^+, v_i'^-)$ ($i = 1, 2, \dots, n$) be two collections of DHBFEs. If $p_i \leq p'_i$ for all i , then

$$\text{DHBFWGHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) \leq \text{DHBFWGHM}^{\rho, \psi}(p'_1, p'_2, \dots, p'_n). \quad (33)$$

Theorem 16 (Boundedness) Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs, and let $p^- = \min_i p_i, p^+ = \max_i p_i$, then

$$p^- \leq \text{DHBFWGHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) \leq p^+. \quad (34)$$

4 Models for MADM with Dual Hesitant Bipolar Fuzzy Information

In this section, we shall utilize the dual hesitant bipolar fuzzy Heronian Mean aggregation operators to multiple attribute decision making with DHBFI information.

For a MADM problem, suppose $A = \{A_1, A_2, \dots, A_m\}$ ($i = 1, 2, \dots, m$) is the set of alternatives, let $C = \{C_1, C_2, \dots, C_n\}$ be the set of attributes. Let $w = (w_1, w_2, \dots, w_n)$ be the weight vector of attributes, where $w_j \geq 0$, $j = 1, 2, \dots, n$, $\sum_{j=1}^n w_j = 1$. Suppose that $P = (p_{ij})_{m \times n} = \langle \{u_{ij}^+\}, \{v_{ij}^-\} \rangle_{m \times n}$ ($i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$) is the dual hesitant bipolar fuzzy decision matrix, where u_{ij}^+ indicates the degree that the alternative A_i satisfies the attribute C_j given by the decision maker, v_{ij}^- indicates the degree that the alternative A_i does not satisfies the attribute C_j given by the decision maker, $u_{ij}^+ \in [0, 1]$, $v_{ij}^- \in [0, 1]$, $(u_{ij}^+)^q + (v_{ij}^-)^q \leq 1$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$.

In the following, we apply the GDHBFWHM (DHBFWGHM) operator to the multiple attribute decision making with DHBFEs information.

Step 1 We utilize the decision information given in matrix P , and the GDHBFWHM operator

$$\begin{aligned} & \text{GDHBFWHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) \\ &= \left\langle \left\{ \left(1 - \prod_{j=1}^n (1 - (\gamma_i^+)^{\rho} (\gamma_j^+)^{\psi})^{w_i w_j} \right)^{\frac{1}{\rho + \psi}} \right\}, \left\{ -1 + \left(1 - \prod_{j=1}^n (1 - (1 + \eta_i^-)^{\rho} (1 + \eta_j^-)^{\psi})^{w_i w_j} \right)^{\frac{1}{\rho + \psi}} \right\} \right\rangle \\ & (i = 1, 2, \dots, m, j = 1, 2, \dots, n) \end{aligned}$$

Or DHBFWGHM operator:

$$\begin{aligned} & \text{DHBFWGHM}^{\rho, \psi}(p_1, p_2, \dots, p_n) \\ &= \left\langle \left\{ 1 - \left(1 - \prod_{j=1}^n (1 - (1 - \gamma_i^+)^{\rho} (1 - \gamma_j^+)^{\psi})^{w_i w_j} \right)^{\frac{1}{\rho + \psi}} \right\}, \left\{ - \left(1 - \prod_{j=1}^n (1 - (-\eta_i^-)^{\rho} (-\eta_j^-)^{\psi})^{w_i w_j} \right)^{\frac{1}{\rho + \psi}} \right\} \right\rangle. \end{aligned}$$

to derive the overall preference values p_i ($i = 1, 2, \dots, m$) of the alternative A_i .

Step 2 Calculate the scores $S(p_i)$ ($i = 1, 2, \dots, m$) of the overall DHBFEs p_i ($i = 1, 2, \dots, m$) to rank all the alternatives A_i ($i = 1, 2, \dots, m$) and then to select the best one(s).

Step 3 Rank all the alternatives A_i ($i = 1, 2, \dots, m$) and select the best one(s) in accordance with $S(p_i)$ ($i = 1, 2, \dots, m$).

Step 4 End.

The flowchart of the proposed algorithm is given in Figure 1.

5 Numerical Example and Comparative Analysis

5.1 Numerical Example

In this section, a numerical example is presented to show potential evaluation of quality of digital government services with dual hesitant bipolar fuzzy information in order to illustrate the method proposed in this paper.

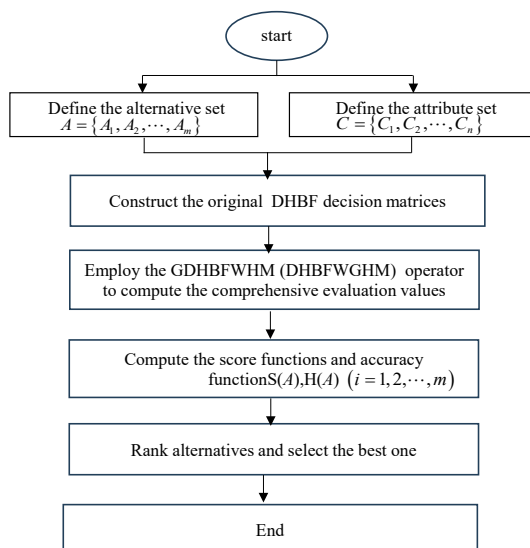


Figure 1 The flowchart of our proposed MAGDM method

In recent years, with “internet plus government service” as the starting point, the “run at most once” reform as the overall traction, and the integrated data platform as the key support, Chongqing has achieved remarkable results in government digital governance, which is very representative. In addition, Zhejiang Province has achieved significant results in economic development and political reform due to its geographical advantage, and has a certain exemplary role in the country. Therefore, this study takes Chongqing as an example to comprehensively evaluate the quality of its digital government services.

There are five possible Chongqing government affairs WeChat official accounts (Chongqing Medical, Chongqing Transportation, Chongqing Weather, Chongqing Public Security Bureau, Chongqing Taxation) A_i ($i = 1, 2, 3, 4, 5$) to select. The expert selects four attribute to evaluate the five possible official accounts: C_1 is the platform design quality; C_2 is the information and administrative services; C_3 the information technology; C_4 is the government WeChat information environment. The weighting vector is $\mathbf{w} = (0.2, 0.1, 0.3, 0.4)^T$. The five possible emerging technology accounts A_i ($i = 1, 2, 3, 4$) are to be evaluated using the dual hesitant bipolar fuzzy information by the decision maker under the above four attributes, and the dual hesitant bipolar fuzzy decision matrix $\mathbf{P} = (\mathbf{p}_{ij})_{4 \times 4} = \langle \{\mathbf{u}_{ij}^+\}, \{\mathbf{v}_{ij}^-\} \rangle_{4 \times 4}$ is shown in Table 1.

Table 1 Dual hesitant bipolar fuzzy set

	C_1	C_2	C_3	C_4
A_1	$\langle \{0.4, 0.5\}, \{-0.8\} \rangle$	$\langle \{0.6, 0.7\}, \{-0.3\} \rangle$	$\langle \{0.2, 0.3\}, \{-0.6\} \rangle$	$\langle \{0.4, 0.5\}, \{-0.7\} \rangle$
A_2	$\langle \{0.6, 0.7\}, \{-0.4, -0.5\} \rangle$	$\langle \{0.7, 0.8\}, \{-0.2\} \rangle$	$\langle \{0.7, 0.8\}, \{-0.2, -0.3\} \rangle$	$\langle \{0.4, 0.5\}, \{-0.5\} \rangle$
A_3	$\langle \{0.5, 0.6\}, \{-0.4, -0.5\} \rangle$	$\langle \{0.5\}, \{-0.7\} \rangle$	$\langle \{0.5, 0.6\}, \{-0.2, -0.3\} \rangle$	$\langle \{0.5, 0.6\}, \{-0.3\} \rangle$
A_4	$\langle \{0.8\}, \{-0.1, -0.2\} \rangle$	$\langle \{0.6, 0.7\}, \{-0.3\} \rangle$	$\langle \{0.3, 0.4\}, \{-0.4\} \rangle$	$\langle \{0.5, 0.6\}, \{-0.6, -0.7\} \rangle$

In order to select the most desirable account, we utilize the GDHBFWHM (DHBFWGHM) operator to develop an approach to multiple attribute decision making problems with dual hesitant bipolar fuzzy information, which can be described as following.

Step 1 We utilize the GDHBFWHM operator to aggregate all the dual hesitant bipolar fuzzy information p_{ij} ($i = 1, 2, 3, 4; j = 1, 2, 3, 4$) into the overall dual hesitant bipolar fuzzy values p_i ($i = 1, 2, 3, 4$). Due to the relatively large numbers, the overall assessments are omitted. Here let $\rho = 1, \psi = 1$.

Step 2 Calculate the score values $S(p_i)$ ($i = 1, 2, 3, 4$) of the overall dual hesitant bipolar fuzzy preference values p_i ($i = 1, 2, 3, 4$).

$$s(p_1) = 0.5942, s(p_2) = 0.7457, s(p_3) = 0.7473, s(p_4) = 0.6967.$$

Step 3 Rank all the account A_i ($i = 1, 2, 3, 4$) in accordance with the score value $s(p_i)$ of the overall fuzzy preference value p_i : $A_3 \succ A_2 \succ A_4 \succ A_1$. And thus, the most desirable account is A_3 .

Furthermore, the DHBFWGHM operator is used to calculate comprehensive dual hesitant bipolar fuzzy information shown as follows.

Step 1 We utilize the DHBFWGHM operator to aggregate all the dual hesitant bipolar fuzzy information p_{ij} ($i = 1, 2, 3, 4; j = 1, 2, 3, 4$) into the overall dual hesitant bipolar fuzzy values p_i ($i = 1, 2, 3, 4$). Due to the relatively large numbers, the overall assessments are omitted. Here let $\rho = 1, \psi = 1, q = 3$.

Step 2 Calculate the score values $s(p_i)$ ($i = 1, 2, 3, 4$) of the overall dual hesitant bipolar fuzzy preference values p_i ($i = 1, 2, 3, 4$).

$$s(p_1) = 0.4835, s(p_2) = 0.6350, s(p_3) = 0.6366, s(p_4) = 0.5860.$$

Step 3 Rank all the account A_i ($i = 1, 2, 3, 4$) in accordance with the score value $s(p_i)$ of the overall fuzzy preference value p_i : $A_3 \succ A_2 \succ A_4 \succ A_1$. And thus, the most desirable account is A_3 .

5.2 The Influence of Parameter on the Ranking Result

In the previous subsection, we only considered the case of the parameter $\rho = 1, \psi = 1$. To discuss the sensitivity and flexibility of the GDHBFWHM (DHBFWGHM) operator to aggregate the evaluation values by changing the value of q, ρ and ψ . The score function values and ranking order of countries are displayed in Table 2 and 3.

From the Table 2, we can see that the ranking order of countries based on GDHBFWHM operator is different for a given value of parameter ρ and ψ . When $\rho = \psi = 1$ and $\rho = \psi = 2$, the ranking order is $A_3 \succ A_2 \succ A_4 \succ A_1$. When $\rho = \psi = 3, 4, 5, 6, 7, 8, 9, 10$, the ranking order is $A_4 \succ A_2 \succ A_3 \succ A_1$. From the Table 3, we can see that the ranking order of countries based on DHBFWGHM operator is different for a given value of parameter ρ and ψ . When $\rho = \psi = 1$, the ranking order is $A_3 \succ A_2 \succ A_4 \succ A_1$. When $\rho = \psi = 2, 3, 4, 5, 6, 7, 8, 9, 10$, the ranking

Table 2 Scores and ranking results by using the GDHBFWMH operator

Parameters	Score function	Ranking results
$(\rho = 1, \psi = 1)$	$s(p_1) = 0.5942, s(p_2) = 0.7457, s(p_3) = 0.7473, s(p_4) = 0.6967.$	$A_3 \succ A_2 \succ A_4 \succ A_1$
$(\rho = 2, \psi = 2)$	$s(p_1) = 0.5843, s(p_2) = 0.7111, s(p_3) = 0.7262, s(p_4) = 0.6858.$	$A_3 \succ A_2 \succ A_4 \succ A_1$
$(\rho = 3, \psi = 3)$	$s(p_1) = 0.5804, s(p_2) = 0.6654, s(p_3) = 0.6568, s(p_4) = 0.6727.$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(\rho = 4, \psi = 4)$	$s(p_1) = 0.5772, s(p_2) = 0.6373, s(p_3) = 0.6134, s(p_4) = 0.6580.$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(\rho = 5, \psi = 5)$	$s(p_1) = 0.5615, s(p_2) = 0.6253, s(p_3) = 0.6080, s(p_4) = 0.6460.$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(\rho = 6, \psi = 6)$	$s(p_1) = 0.5332, s(p_2) = 0.6189, s(p_3) = 0.5923, s(p_4) = 0.6344.$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(\rho = 7, \psi = 7)$	$s(p_1) = 0.5088, s(p_2) = 0.6091, s(p_3) = 0.5842, s(p_4) = 0.6264.$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(\rho = 8, \psi = 8)$	$s(p_1) = 0.4957, s(p_2) = 0.5811, s(p_3) = 0.5765, s(p_4) = 0.6149.$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(\rho = 9, \psi = 9)$	$s(p_1) = 0.4822, s(p_2) = 0.5737, s(p_3) = 0.5553, s(p_4) = 0.6066.$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(\rho = 10, \psi = 10)$	$s(p_1) = 0.4756, s(p_2) = 0.5609, s(p_3) = 0.5211, s(p_4) = 0.5943.$	$A_4 \succ A_2 \succ A_3 \succ A_1$

Table 3 Scores and ranking results by using the DHBFWGHM operator

Parameters	Score function	Ranking results
$(\rho = 1, \psi = 1)$	$s(p_1) = 0.4835, s(p_2) = 0.6350, s(p_3) = 0.6366, s(p_4) = 0.5860.$	$A_3 \succ A_2 \succ A_4 \succ A_1$
$(\rho = 2, \psi = 2)$	$s(p_1) = 0.4436, s(p_2) = 0.6004, s(p_3) = 0.5898, s(p_4) = 0.5461.$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$(\rho = 3, \psi = 3)$	$s(p_1) = 0.4208, s(p_2) = 0.5823, s(p_3) = 0.5571, s(p_4) = 0.5250.$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$(\rho = 4, \psi = 4)$	$s(p_1) = 0.4044, s(p_2) = 0.5702, s(p_3) = 0.5350, s(p_4) = 0.5115.$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$(\rho = 5, \psi = 5)$	$s(p_1) = 0.3916, s(p_2) = 0.5611, s(p_3) = 0.5194, s(p_4) = 0.5021.$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$(\rho = 6, \psi = 6)$	$s(p_1) = 0.3854, s(p_2) = 0.5567, s(p_3) = 0.5077, s(p_4) = 0.4942.$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$(\rho = 7, \psi = 7)$	$s(p_1) = 0.3737, s(p_2) = 0.5482, s(p_3) = 0.4919, s(p_4) = 0.4858.$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$(\rho = 8, \psi = 8)$	$s(p_1) = 0.3659, s(p_2) = 0.5338, s(p_3) = 0.4836, s(p_4) = 0.4762.$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$(\rho = 9, \psi = 9)$	$s(p_1) = 0.3518, s(p_2) = 0.5277, s(p_3) = 0.4747, s(p_4) = 0.4619.$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$(\rho = 10, \psi = 10)$	$s(p_1) = 0.3469, s(p_2) = 0.5147, s(p_3) = 0.4605, s(p_4) = 0.4521.$	$A_2 \succ A_3 \succ A_4 \succ A_1$

order is $A_2 \succ A_3 \succ A_4 \succ A_1$. Tables 2 and 3 show that the ranking results increase and become steady with the increase of values in parameter vector. That is, the results become increasingly objective by considering the interrelationship among the attribute values.

It can be easily seen from the above analysis that although different aggregation operators are used, the overall ranking of the alternative solutions will slightly different. However, the most desirable solution is still A_2 or A_4 . This is because the personal preferences, values, and experiences of decision-makers can affect their evaluation and ranking of solutions. Different decision-makers may have different views and judgments on the same plan, and in reality, decision-makers can choose different plans based on their preferences.

5.3 Comparative Analysis

To elaborate the effectiveness and practicability of the proposed approach, we give a comparative analysis of the developed operators with several previous decision methods, including the dual hesitant bipolar fuzzy weighted average (DHBFWA) operator and dual hesitant bipolar fuzzy weighted geometric (DHBFWG) operator proposed by Xu et al.^[48]. We will discuss this example in the following parts.

Definition 16^[48] Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs, and $\mathbf{w} = (w_1, w_2, \dots, w_n)^T$ be the weight vector of $\mathbf{p}_i$ ($i = 1, 2, \dots, n$) with $w_i \geq 0$ ($i = 1, 2, \dots, n$) where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. The dual hesitant bipolar fuzzy weighted average (DHBFWA) operator is a mapping DHBFWA:

DHBFEⁿ $\rightarrow$ DHBFE can be defined as

$$\text{DHBFWA}(p_1, p_2, \dots, p_n) = \bigoplus_{i=1}^n w_i p_i = \left\langle \left\{ 1 - \prod_{i=1}^n (1 - (\gamma_i^+)^{w_i}) \right\}, \left\{ -\prod_{i=1}^n (\eta_j^{-w_i}) \right\} \right\rangle. \quad (35)$$

Definition 17^[48] Let $p_i = (u_i^+, v_i^-)$ ($i = 1, 2, \dots, n$) be a collection of DHBFEs, and $\mathbf{w} = (w_1, w_2, \dots, w_n)^T$ be the weight vector of $\mathbf{p}_i$ ($i = 1, 2, \dots, n$) with $w_i \geq 0$ ($i = 1, 2, \dots, n$) where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. The dual hesitant bipolar fuzzy weighted geometric (DHBFWG) operator is a mapping DHBFWG:

DHBFEⁿ $\rightarrow$ DHBFE can be defined as

$$\text{DHBFWG}(p_1, p_2, \dots, p_n) = \bigotimes_{i=1}^n p_i^{w_i} = \left\langle \left\{ \prod_{i=1}^n (\gamma_j^{+w_i}) \right\}, \left\{ -1 + \prod_{i=1}^n (1 + (\eta_i^-)^{w_i}) \right\} \right\rangle. \quad (36)$$

By using the decision information given in this example, and the DHBFWA and DHBFWG operators, the aggregating results are shown in follows.

Step 1. We utilize the DHBFWA operator to aggregate all the dual hesitant bipolar fuzzy information p_{ij} ($i = 1, 2, 3, 4; j = 1, 2, 3, 4$) into the overall dual hesitant bipolar fuzzy values p_i ($i = 1, 2, 3, 4$).

Step 2. Calculate the score values $S(p_i)$ ($i = 1, 2, 3, 4$) of the overall dual hesitant bipolar fuzzy preference values p_i ($i = 1, 2, 3, 4$).

$$s(p_1) = 0.6053, s(p_2) = 0.8535, s(p_3) = 0.7021, s(p_4) = 0.8597.$$

Step 3. Rank all the account A_i ($i = 1, 2, 3, 4$) in accordance with the score value $s(p_i)$ of the overall fuzzy preference value p_i : $A_4 \succ A_2 \succ A_3 \succ A_1$. And thus, the most desirable account is A_4 .

Furthermore, the DHBFWG operator is used to calculate comprehensive dual hesitant bipolar fuzzy information shown as follows.

Step 1. We utilize the DHBFWG operator to aggregate all the dual hesitant bipolar fuzzy information p_{ij} ($i = 1, 2, 3, 4; j = 1, 2, 3, 4$) into the overall dual hesitant bipolar fuzzy values p_i ($i = 1, 2, 3, 4$).

Step 2. Calculate the score values $s(p_i)$ ($i = 1, 2, 3, 4$) of the overall dual hesitant bipolar fuzzy preference values p_i ($i = 1, 2, 3, 4$).

$$s(p_1) = 0.5705, s(p_2) = 0.7358, s(p_3) = 0.7265, s(p_4) = 0.6989.$$

Step 3. Rank all the account A_i ($i = 1, 2, 3, 4$) in accordance with the score value $s(p_i)$ of the overall fuzzy preference value p_i : $A_2 \succ A_3 \succ A_4 \succ A_1$. And thus, the most desirable account is A_2 .

In comparison to the methods introduced by Xu et al.^[48], our proposed GDHBFWHM operators and DHBFWGHM operators proposed in this paper can reflect the interrelationship between the attributes, especially the partition structure between the attributes as mentioned in the above example. This is the reason why the results obtained by the DHBFWA operator and DHBFWG operator are significantly different from those obtained by the proposed GDHBFWHM operator and DHBFWGHM operators. Thus, compared with the DHBFWA operator and DHBFWG operator, the proposed GDHBFWHM operator and DHBFWGHM operator are more suitable for modeling such decision problems.

6 Conclusion

6.1 The Findings of This Study

In this paper, a novel MADM approach based on dual hesitant bipolar fuzzy sets (DHBFS) is proposed. And some properties, operational laws, score and accuracy functions of DHBFS are discussed. Considering that the advantage of Heronian Mean operator in the aggregation process due to the parameter involved, and the use of the Heronian Mean operator provides flexibility in capturing the interactions between any number of attributes with every possible permutation, this paper developed some novel Heronian Mean aggregation operators under DHBFS environment. The main findings of this study are as follows:

- 1) The concept of the DHBFS was proposed by combining the bipolar fuzzy sets (BFSs) and hesitant fuzzy set (HFS).
- 2) Some DHBFS aggregation operators were proposed, such as the generalized dual hesitant bipolar fuzzy Heronian Mean (GDHBFHM) operator, the generalized dual hesitant bipolar fuzzy weighted Heronian Mean (GDHBFWHM) operator, the dual hesitant bipolar fuzzy geometric Heronian Mean (DHBFGHM) operator, the dual hesitant bipolar fuzzy weighted geometric Heronian Mean (DHBFWGHM) operator.
- 3) An MADM method based on the proposed operators was developed, and a numerical example was illustrated to demonstrate the effectiveness and superiority of the proposed method through comparison with existing methods.

6.2 The Advantages and Disadvantages of the Proposed Method

The advantages of the proposed method is as follows.

- 1) Consider the interrelationships between attributes: The proposed operator can handle the dependencies and correlations between attributes, which is particularly important when dealing with multi-attribute decision-making problems. It can better reflect the actual situation and improve the accuracy of decision-making.
- 2) Dealing with complex decision-making environments: As the uncertainty of decision information and the complexity of the decision-making environment increase, the proposed operator can effectively integrate and handle these complexities, providing more reasonable decision-making basis.

3) Expanded the constraints of membership and non-membership: The proposed operator performs well in the field of fuzzy decision-making, especially when dealing with Fermat hesitant fuzzy information, providing effective decision-making methods.

The disadvantages of the proposed method

1) Computational complexity: The computation of proposed operators may be relatively complex, especially when dealing with large amounts of data and highly complex decision environments, which may require high computational resources and time.

2) Limited applicability: Although the proposed operator performs well in handling multi-attribute decision problems, it may not be suitable for all types of decision problems. The scope and conditions of its application need to be analyzed on a case-by-case basis.

In summary, the proposed operator has significant advantages in multi-attribute decision-making, especially when dealing with complex problems where there are dependencies and correlations between attributes. However, the computational complexity and limitations of its applicability are also factors that need to be considered. In practical applications, the decision to use the Heronian Mean operator and how to use it should be based on the specific characteristics and requirements of the problem.

6.3 The Future Research

Although the proposed operators theoretically have the advantage of handling multi-attribute decision problems, there are still challenges in effectively applying these theories to practical problems, especially when dealing with decision information with priority and uncertainty. This requires future research to not only focus on theoretical innovation, but also strengthen practical application research to narrow the gap between theory and practice.

Future research will be directed towards addressing this limitation and enhancing the method's capabilities in the following ways:

1) Expanding application scope: Future research should focus on applying Heronian Mean operators to a wider range of decision information to meet a wider range of decision-making needs. This will help improve the scientificity and accuracy of decision-making.

2) Strengthening practical application research: Future research should pay more attention to the application of Heronian Mean operators in practical problems, especially when dealing with decision information with priority and uncertainty. By studying specific cases, we can better understand the application conditions and limitations of Heronian Mean operators, and provide guidance for their practical application in real-world problems.

3) Combining new technologies and methods: With the development of technology, future research can explore the integration of Heronian Mean operators with other new technologies and methods, such as big data analysis techniques, machine learning algorithms, etc., to improve the efficiency and accuracy of decision-making. This will help broaden the application areas of Heronian Mean operators and enable them to play a greater role in solving complex decision-making problems.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict of Interest The authors declare no conflict of interest.

References

- [1] Zadeh L A. Fuzzy sets. *Information and Control*, 1965, 8(3): 338–353.
- [2] Pei Z, Liu J, Hao F, et al. FLM-TOPSIS: The fuzzy linguistic multiset TOPSIS method and its application in linguistic decision making. *Information Fusion*, 2019, 45: 266–281.
- [3] Kong M, Pei Z, Ren F, et al. New operations on generalized hesitant fuzzy linguistic term sets for linguistic decision making. *International Journal of Fuzzy Systems*, 2019, 21(1): 243–262.
- [4] Liu Y, Liu J, Qin Y. Pythagorean fuzzy linguistic Muirhead Mean operators and their applications to multiattribute decision-making. *International Journal of Intelligent Systems*, 2020, 35(2): 300–332.
- [5] Liu P, Chen S M, Wang P. Multiple-attribute group decision-making based on q-rung orthopair fuzzy power maclaurin symmetric Mean operators. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 2018, 50(10): 3741–3756.
- [6] Seiti H, Hafezalkotob A, Martinez L. R-sets, comprehensive fuzzy sets risk modeling for risk-based information fusion and decision-making. *IEEE Transactions on Fuzzy Systems*, 2021, 29(2): 385–399.
- [7] Liu W S, Liao H C. A bibliometric analysis of fuzzy decision research during 1970–2015. *International Journal of Fuzzy Systems*, 2017, 19: 1–14.
- [8] Turksen L B. Interval valued fuzzy sets based on normal forms. *Fuzzy Sets and Systems*, 1980, 80: 191–210.
- [9] Castillo O, Melin P. Optimization of type-2 fuzzy systems based on bio-inspired methods: A concise review. *Information Sciences*, 2012, 205: 1–19.
- [10] Dubois D, Prade H. *Fuzzy sets and systems: Theory and applications*. New York: Academic Press, 1980.
- [11] Atanassov K T. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 1986, 20: 87–96.
- [12] Torra V. Hesitant fuzzy sets. *International Journal of Intelligent Systems*, 2010, 25(6): 529–539.
- [13] Tao Y F, Peng Y, Wu Y H. Linguistic dual hesitant fuzzy preference relations and their application in group decision-making. *International Journal of Fuzzy Systems*, 2023, 25(3): 1105–1130.
- [14] Liu P P, Wang H J, Ning B Q, et al. MAGDM method based on generalized hesitant fuzzy TODIM and cumulative prospect theory and application to recruitment of university researchers. *Journal of Intelligent & Fuzzy Systems*, 2023, 45(1): 1–18.

- [15] Sun G, Luo J, Wang G J. TOPSIS method based on hesitant factor and priority weighted operator in pythagorean fuzzy environment. *International Journal of Fuzzy Systems*, 2023, 25(2): 831–850.
- [16] Liu Q, Hou J D, Dong Q. Modified MARCOS method for industrial competitiveness evaluation of regional cultural tourism with probabilistic hesitant fuzzy information. *Journal of Intelligent & Fuzzy Systems*, 2023, 45(1): 1–11.
- [17] Wang X D, Xu Z S, Qin Y. A CVaR optimization method for priority of hesitant fuzzy preference relation with chance constraint. *Journal of Intelligent & Fuzzy Systems*, 2023, 44(1): 395–408.
- [18] Naqvi D, Sachdev G, Ahmad I. Matrix games involving interval-valued hesitant fuzzy linguistic sets and its application to electric vehicles. *Journal of Intelligent & Fuzzy Systems*, 2023, 44(3): 5085–5105.
- [19] Wu Z, Liao H C. An approach to hesitant fuzzy linguistic multiple criteria group decision making with uncertain criteria weights considering incomparability between alternatives. *Journal of the Operational Research Society*, 2023, 74(12): 2606–2618.
- [20] Zhang Z X, Wang L, Wang Y M, et al. A novel alpha-level sets based fuzzy DEMATEL method considering experts' hesitant information. *Expert Systems with Applications*, 2023, 213: 118925.
- [21] Meng L Y, Li L Q, Xie W X, et al. Time-sequential hesitant fuzzy entropy, cross-entropy and correlation coefficient and their application to decision making. *Engineering Applications of Artificial Intelligence*, 2023, 123(Part C): 106455.
- [22] Fang B. Some uncertainty measures for probabilistic hesitant fuzzy information. *Information Sciences*, 2023, 625: 255–276.
- [23] Zhang W R. Bipolar fuzzy sets. *Proceedings of FUZZY-IEEE*, 1998: 835–840.
- [24] Xu X R, Wei G W. Dual hesitant bipolar fuzzy aggregation operators in multiple attribute decision making. *International Journal of Knowledge-based and Intelligent Engineering Systems*, 2017, 21: 155–164.
- [25] Wei G W, Alsaadi F E, Hayat T, et al. Hesitant bipolar fuzzy aggregation operators in multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, 2017, 33: 1119–1128.
- [26] Jana C, Pal M, Wang J Q. Bipolar fuzzy Dombi prioritized aggregation operators in multiple attribute decision making. *Soft Computing*, 2020, 24: 3631–3646.
- [27] Wei G W, Alsaadi F E, Hayat T, et al. Bipolar fuzzy hamacher aggregation operators in multiple attribute decision making. *International Journal of Fuzzy Systems*, 2018, 20(1): 1–12.
- [28] Gao H, Wei G W, Huang Y H. Dual hesitant bipolar fuzzy hamacher prioritized aggregation operators in multiple attribute decision making. *IEEE Access*, 2018.
- [29] Riaz M, Tehrim S T. Cubic bipolar fuzzy set with application to multi-criteria group decision making using geometric aggregation operators. *Soft Computing*, 2020, 24: 16111–16133.
- [30] Gao H, Lua M, Wei Y. Dual hesitant bipolar fuzzy hamacher aggregation operators and their applications to multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, 2019, 37: 5755–5766.
- [31] Beliakov G, Pradera A, Calvo T. *Aggregation functions: A guide for practitioners*. Berlin: Springer, 2007.
- [32] Tahir M, Ubaid U R, Muhammad N. A novel approach towards Heronian Mean operators in multiple attribute decision making under the environment of bipolar complex fuzzy information. *AIMS Mathematics*, 2023, 8(1): 1848–1870.
- [33] Qiyas M, Abdullah S, Khan S. A novel approach of linguistic picture fuzzy dombi Heronian Mean

- operators and their application to emergency program selection. *Journal of Experimental & Theoretical Artificial Intelligence*, 2023, 35(3): 445–472.
- [34] Rao F S, Xiao M Y. A novel MADM algorithm for physical education teaching quality evaluation based on 2-tuple linguistic neutrosophic numbers power Heronian Mean operators. *PloS One*, 2023, 18(2): e0279534.
- [35] Akram M, Naz S, Feng F, et al. Extended MABAC method based on 2-tuple linguistic T-spherical fuzzy sets and Heronian Mean operators: An application to alternative fuel selection. *AIMS Mathematics*, 2023, 8(5): 10619–10653.
- [36] Xu J, Ma Z M, Xu Z S. Generalized cross weighted Heronian Mean operators and their application to binary classification. *Journal of Intelligent and Fuzzy Systems*, 2023, 44(2): 2779–2789.
- [37] Anusha V, Sireesha V. Einstein Heronian Mean aggregation operator and its application in decision making problems. *Computational and Applied Mathematics*, 2022, 41(2): 69.
- [38] Tian Y B, Ma Z M. Covering-based compound mean operators arising from Heronian and Bonferoni Mean operators in fuzzy and intuitionistic fuzzy environments. *Journal of Intelligent & Fuzzy Systems*, 2022, 42(3): 2115–2126.
- [39] Fan J P, Zhang H, Wu M Q. Dynamic multi-attribute decision-making based on interval-valued picture fuzzy geometric Heronian Mean operators. *IEEE Access*, 2022, 10: 12070–12083.
- [40] Kumar K, Chen S M. Group decision making based on advanced intuitionistic fuzzy weighted Heronian Mean aggregation operator of intuitionistic fuzzy values. *Information Sciences*, 2022, 601: 306–322.
- [41] Hazwani H, Harish G, Ashraf A Q, et al. Heronian Mean operators considering shapley fuzzy measure under interval neutrosophic vague environment for an investment decision. *International Journal of Fuzzy Systems*, 2022, 24(4): 2068–2091.
- [42] Tahir M, Izatmand, Ali Z, et al. A method to multi-attribute decision making problems by using Heronian Mean operators based on linear diophantine uncertain linguistic settings. *Journal of Intelligent & Fuzzy Systems*, 2022, 42(6): 5291–5319.
- [43] Zhang H Y, Wei G W, Chen X D. Spherical fuzzy Dombi power Heronian Mean aggregation operators for multiple attribute group decision-making. *Computational and Applied Mathematics*, 2022, 41(3): 98.
- [44] Wang H L, Zhang F M. Interaction power Heronian Mean aggregation operators for multiple attribute decision making with T-spherical fuzzy information. *Journal of Intelligent and Fuzzy Systems*, 2022, 42(6): 5715–5739.
- [45] Wang H D, Zhang Y, Bai C Z. A new three-way group decision-making model based on geometric Heronian Mean operators with q-rung orthopair uncertain linguistic information. *Journal of Intelligent & Fuzzy Systems*, 2022, 43(1): 1–20.
- [46] Torra V. Hesitant fuzzy sets. *International Journal of Intelligent Systems*, 2010, 25(6): 529–539.
- [47] Yu D J. Intuitionistic fuzzy geometric Heronian Mean aggregation operators. *Applied Soft Computing*, 2012, 13: 1235–1246.
- [48] Xu X R, Wei G W. Dual hesitant bipolar fuzzy aggregation operators in multiple attribute decision making. *International Journal of Knowledge-based and Intelligent Engineering Systems*, 2017, 21: 155–164.